ON THE EVALUATION OF CERTAIN SINGULAR INTEGRALS WITH A KERNEL OF THE CAUCHY TYPE

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In the solution of many problems of hydro-aerodynamics [1,2] and in the theories of elasticity [3,4] and filtration [5,6], there occur integrals with kernels of the Cauchy type defined on intervals of the real axis. Some of these integrals can be reduced by means of simple transformations to either one of two integrals of the following type:

$$J(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{t-x}\,dt \qquad (-1 \leqslant x \leqslant 1) \tag{0.1}$$

$$I(x) = \frac{\sqrt{1-x^2}}{\pi}\int_{-1}^{1}\frac{f(t)}{t-x}\,\frac{dt}{\sqrt{1-t^2}} \qquad (-1 \leqslant x \leqslant 1) \tag{0.2}$$

These integrals exist in the sense of Cauchy's principal value if the function $f(x)$ satisfies Hoelder's condition [7,8] on the interval between the limits -1 and 1.

With the aid of the transformation of variables $x = \cos\theta$, $t = \cos\phi$, the integrals $J(x)$, $I(x)$ can be written in the trigonometric form

$$J(\cos\theta) = \frac{1}{\pi}\int_{0}^{\pi}\frac{f(\cos\varphi)}{\cos\varphi - \cos\theta}\sin\varphi\,d\varphi \qquad (0 \leqslant \theta \leqslant \pi) \tag{0.3}$$

$$I(\cos\theta) = \frac{\sin\theta}{\pi}\int_{0}^{\pi}\frac{f(\cos\varphi)}{\cos\varphi - \cos\theta}\,d\varphi \qquad (0 \leqslant \theta \leqslant \pi) \tag{0.4}$$

In this form these integrals appear in the theory of wings [2,9]. There exist many types of integrals which can be transformed to one of the integrals $J(x)$ or $I(x)$ by means of simple transformations and, hence, to either one of them. For example, the known singular integrals whose kernels are either the cotangent or logarithm, and which are met in the theory of waves and jets, can be reduced to the integrals $J(x)$ and $I(x)$ be means of the relations

$$\frac{1}{\pi}\int_0^\pi f(\cos\varphi)\operatorname{ctg}\frac{\theta-\varphi}{2}\,d\varphi = J(\cos\theta) + I(\cos\theta) \tag{0.5}$$

$$\frac{1}{\pi}\int_0^\pi \frac{d}{d\varphi} f(\cos\varphi)\ln\left|\sin\frac{\varphi-\theta}{2}\csc\frac{\varphi+\theta}{2}\right| d\varphi = J(\cos\theta) \tag{0.6}$$

It is known that, due to the fact that the integrals $J(x)$ and $I(x)$ are singular, one cannot directly apply many formulas of mechanical quadrature [13] for the evaluation of these integrals, even though those formulas may be valid for ordinary Riemann integrals. However, if one takes into account the properties of the integrands, one may with the aid of various transformations obtain various formulas for mechanical quadratures that are applicable to singular integrals also. Such formulas of mechanical quadratures were obtained by Multhoop [9] for the integrals (0.3) and (0.4), by Kalandiia [10] for the integrals (0.1) and (0.2), and by Simonov [11] for the integral (0.5).

Below there are derived formulas which make it possible to obtain approximate expressions for the integrals $J(x)$ and $I(x)$ and to estimate the error in the approximation. The obtained formulas contain functions defined in terms of elementary functions or in terms of rapidly converging series. Tables are given for some of these functions.

1. Let us introduce into our consideration the Chebyshev polynomials of the first and second kind

$$T_n(x) = \cos n \arccos x, \qquad U_n(x) = \sin n \arccos x$$

It is not difficult to show that $T_n(x)$ and $U_n(x)$ satisfy the relations

$$\frac{1}{\pi}\int_{-1}^{1}\frac{U_n(t)}{t-x}dt = -T_n(x), \qquad \frac{\sqrt{1-x^2}}{\pi}\int_{-1}^{1}\frac{T_n(t)}{t-x}\frac{dt}{\sqrt{1-t^2}} = U_n(x) \tag{1.1}$$

Let the function $f(x)$, which appears in the integrals $J(x)$ and $I(x)$, satisfy the Hoelder condition on the interval between the limits -1 and 1. Then the integrals $J(x)$ and $I(x)$ can be represented in the form of the series

$$J(x) = \sum_{n=1}^{\infty} - b_n T_n(x) \qquad (-1 \leqslant x \leqslant 1) \tag{1.2}$$

$$I(x) = \sum_{n=1}^{\infty} a_n U_n(x) \qquad (-1 \leqslant x \leqslant 1) \tag{1.3}$$

where

$$a_n = \frac{2}{\pi}\int_{-1}^{1} f(x)\,T_n(x)\frac{dx}{\sqrt{1-x^2}}, \quad b_n = \frac{2}{\pi}\int_{-1}^{1} f(x)\,U_n(x)\frac{dx}{\sqrt{1-x^2}} \tag{1.4}$$

Indeed, in this case it follows from the theory of the Fourier series that the function $f(x)$ can be represented on the interval between the limits -1 and 1 by means of a series in terms of Chebyshev polynomials of the first kind

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n T_n(x) \qquad (-1 \leqslant x \leqslant 1) \tag{1.5}$$

or in the form of a series in polynomials of the second kind

$$f(x) = \sum_{n=1}^{\infty} b_n U_n(x) \qquad (-1 \leqslant x \leqslant 1) \tag{1.6}$$

Let us replace the function $f(x)$ in the integral $J(x)$ by the series (1.6) and in the integral (1.5) by the series (1.5). With the aid of the relation (1.1) we then obtain the formulas (1.2) and (1.3). The use of these formulas for the computation of the integrals $J(x)$ and $I(x)$ is not recommended, for the series (1.2) and (1.3) converge slowly, and the coefficients a_n and b_n must be evaluated by methods of numerical integration. In the case when the series appearing on the right-hand sides can be summed in a closed form, the formulas (1.2) and (1.3) give the exact values of the integrals $J(x)$ and $I(x)$. If the integrals $J(x)$ and $I(x)$ are given in the trigonometric forms (0.3) and (0.4), the equations (1.2) and (1.3) take on the form:

$$J(\cos\theta) = -\sum_{n=1}^{\infty} b_n \cos n\theta \qquad (0 \leqslant \theta \leqslant \pi) \tag{1.7}$$

$$I(\cos\theta) = \sum_{n=1}^{\infty} a_n \sin n\theta \qquad (0 \leqslant \theta \leqslant \pi) \tag{1.8}$$

The formulas (1.7) and (1.8) make it possible to use the results of the theory of trigonometric series in the investigation of the properties of the integrals $J(x)$ and $I(x)$.

2. Let us consider the functions defined on the interval between the limits -1 and 1 by the equations

$$p_1^{(s)}(x) = \sum_{n=2}^{\infty} \frac{1}{(2n-1)^s} T_{2n-1}(x), \qquad p_2^{(s)}(x) = \sum_{n=2}^{\infty} \frac{1}{(2n)^s} T_{2n}(x) \tag{2.1}$$

$$(s = 1, 2, \ldots; \ -1 \leqslant x \leqslant 1)$$

$$q_1^{(s)}(x) = \sum_{n=2}^{\infty} \frac{1}{(2n-1)^s} U_{2n-1}(x), \qquad q_2^{(s)}(x) = \sum_{n=2}^{\infty} \frac{1}{(2n)^s} U_{2n}(x) \tag{2.2}$$

$$(s = 1, 2, \ldots; \ -1 \leqslant x \leqslant 1)$$

We shall point out certain properties of these functions. The functions $p_1^{(1)}(x)$ and $p_2^{(1)}(x)$ have singularities of the logarithmic type at the points $x = 1$, and $x = -1$;

$$p_1^{(1)}(x) = \frac{1}{4} \ln \frac{1+x}{1-x} - x, \qquad p_2^{(1)}(x) = -\frac{1}{4} \ln 4(1-x^2) - \frac{1}{2}(2x^2 - 1)$$

while the functions $q_1^{(1)}(x)$ and $q_2^{(1)}(x)$ can be expressed on the interval between the limits -1 and 1 in the form

$$q_1^{(1)}(x) = \frac{1}{4}\pi - \sqrt{1-x^2}, \qquad q_2^{(1)}(x) = \frac{1}{4}\pi - \frac{1}{2} \arccos x - x\sqrt{1-x^2}$$

The remaining functions are continuous on the entire interval between -1 and 1. We note also that $p_2^{(s)}(x)$ and $q_1^{(s)}(x)$ are even functions, while $p_1^{(s)}(x)$ and $q_2^{(s)}(x)$ are odd functions, i.e.

$$\begin{aligned} p_1^{(s)}(-x) &= -p_1^{(s)}(x), \qquad & p_2^{(s)}(-x) &= p_2^{(s)}(x) \\ q_1^{(s)}(-x) &= q_1^{(s)}(x), \qquad & q_2^{(s)}(-x) &= -q_2^{(s)}(x) \end{aligned} \tag{2.3}$$

If one makes use of the equations (1.1), it is easy to show that all the introduced functions satisfy the following integral relations:

$$\frac{\sqrt{1-x^2}}{\pi} \int_{-1}^{1} \frac{p_1^{(s)}(t)}{t-x} \frac{dt}{\sqrt{1-t^2}} = q_1^{(s)}(x), \qquad \frac{\sqrt{1-x^2}}{\pi} \int_{-1}^{1} \frac{p_2^{(s)}(t)}{t-x} \frac{dt}{\sqrt{1-t^2}} = q_2^{(s)}(x)$$

$$\frac{\sqrt{1-x^2}}{\pi} \int_{-1}^{1} \frac{(\operatorname{sign} t)\, q_1^{(s)}(t^*)}{t-x} \frac{dt}{\sqrt{1-t^2}} = p_1^{(s)}(x^*)$$

$$\frac{\sqrt{1-x^2}}{\pi} \int_{-1}^{1} \frac{p_2^{(s)}(t^*)}{t-x} \frac{dt}{\sqrt{1-t^2}} = -(\operatorname{sign} x)\, q_2^{(s)}(x^*)$$

$$\frac{1}{\pi} \int_{-1}^{1} \frac{q_1^{(s)}(t)}{t-x} dt = -p_1^{(s)}(x), \qquad \frac{1}{\pi} \int_{-1}^{1} \frac{q_2^{(s)}(t)}{t-x} dt = -p_2^{(s)}(x) \tag{2.4}$$

$$\frac{1}{\pi} \int_{-1}^{1} \frac{p_1^{(s)}(t^*)}{t-x} dt = -(\operatorname{sign} x)\, q_1^{(s)}(x^*), \qquad \frac{1}{\pi} \int_{-1}^{1} \frac{(\operatorname{sign} t)\, q_2^{(s)}(t^*)}{t-x} dt = p_2^{(s)}(x^*) \tag{2.5}$$

where

$$x^* = \sqrt{1-x^2}, \qquad t^* = \sqrt{1-t^2} \tag{2.6}$$

The equations (2.3) make it possible to restrict the computation and study of the properties of the introduced functions to the interval between the limits 0 and 1. By means of a change of variables $x = \cos\theta$, the functions $p_1^{(s)}(x)$, $p_2^{(s)}(x)$, $q_1^{(s)}(x)$ can be rewritten in the trigonometric forms:

$$p_1^{(s)}(\cos\theta) = \sum_{n=2}^{\infty} \frac{\cos(2n-1)\theta}{(2n-1)^s}, \qquad p_2^{(s)}(\cos\theta) = \sum_{n=2}^{\infty} \frac{\cos 2n\theta}{(2n)^s}$$
$$(s = 1, 2, \ldots;\ 0 \leqslant \theta \leqslant \pi) \tag{2.7}$$

$$q_1^{(s)}(\cos\theta) = \sum_{n=2}^{\infty} \frac{\sin(2n-1)\theta}{(2n-1)^s}, \qquad q_2^{(s)}(\cos\theta) = \sum_{n=2}^{\infty} \frac{\sin 2n\theta}{(2n)^s}$$
$$(s = 1, 2, \ldots;\ 0 \leqslant \theta \leqslant \pi) \tag{2.8}$$

Such a form is convenient in the computation of these functions, since in this one does not have to deal with Chebyshev polynomials but with trigonometric functions. In the attached Tables 1–3, the values of some of the functions considered are given, computed with a precision of four places. Tables 1 and 2 were computed with the aid of formulas (2.1) and (2.2); Table 3 was computed using formulas (2.7) and (2.8).

TABLE 1.

x	$p_1^{(1)}(x)$	$p_1^{(3)}(x)$	$p_2^{(1)}(x)$	$p_2^{(3)}(x)$	$q_1^{(2)}(x)$	$q_1^{(4)}(x)$	$q_2^{(2)}(x)$	$q_2^{(4)}(x)$
0.0	0.0000	0.0000	0.1534	0.0123	−0.0840	−0.0111	0.0000	0.0
0.1	−0.0498	−0.0083	0.1459	0.0115	−0.0815	−0.0106	−0.0151	−0.0012
0.2	−0.0986	−0.0162	0.1236	0.0093	−0.0740	−0.0095	−0.0289	−0.0023
0.3	−0.1452	−0.0233	0.0870	0.0058	−0.0614	−0.0073	−0.0399	−0.0031
0.4	−0.1882	−0.0289	0.0370	0.0010	−0.0435	−0.0045	−0.0466	−0.0034
0.5	−0.2253	−0.0325	−0.0246	−0.0043	−0.0202	−0.0011	−0.0473	−0.0032
0.6	−0.2534	−0.0333	−0.0950	−0.0096	0.0086	0.0029	−0.0401	−0.0024
0.7	−0.2663	−0.0299	−0.1682	−0.0139	0.0431	0.0071	−0.0227	−0.0008
0.8	−0.2507	−0.0203	−0.2312	−0.0151	0.0829	0.0110	0.0080	0.0014
0.9	−0.1639	−0.0001	−0.2414	−0.0091	0.1242	0.0133	0.0582	0.0039
1.0	∞	0.0518	∞	0.0253	∞	0.0000	0,0000	0.0000

TABLE 2.

x	$p_1^{(1)}(x^*)$	$p_1^{(3)}(x^*)$	$p_2^{(1)}(x^*)$	$p_2^{(3)}(x^*)$	$q_1^{(2)}(x^*)$	$q_1^{(4)}(x^*)$	$q_2^{(2)}(x^*)$	$q_2^{(4)}(x^*)$
0.0	∞	0.0518	∞	0.0253	0.0000	0.0000	0.0000	0.0000
0.1	0.5016	0.0455	0.3147	0.0200	0.0999	0.0050	0.0809	0.0023
0.2	0.1664	0.0335	—0.0018	0.0108	0.1317	0.0090	0.0945	0.0039
0.3	—0.0170	0.0194	—0.1546	0.0014	0.1386	0.0117	0.0856	0.0045
0.4	—0.1331	0.0049	—0.2284	—0.0067	0.1301	0.0130	0.0645	0.0042
0.5	—0.2075	0.0086	—0.2500	—0.0124	0.1106	0.0128	0.0372	0.0031
0.6	—0.2507	—0.0203	—0.2312	—0.0151	0.0829	0.0110	0.0080	0.0014
0.7	—0.2663	—0.0290	—0.1782	—0.0143	0.0484	0.0077	—0.0192	—0.0005
0.8	—0.2534	—0.0333	—0.0950	—0.0096	0.0086	0.0029	—0.0402	—0.0024
0.9	—0.2023	—0.0305	0.0161	—0.0008	—0.0358	—0.0034	—0.0476	—0.0034
1.0	0.0000	0.0000	0.1534	0.0123	—0.0840	—0.0111	0.0000	0.0000

TABLE 3

x	$p_1^{(1)}(x)$	$p_1^{(3)}(x)$	$p_2^{(1)}(x)$	$p_2^{(3)}(x)$	$q_1^{(2)}(x)$	$q_1^{(4)}(x)$	$q_2^{(2)}(x)$	$q_2^{(4)}(x)$
cos 0°	∞	0.0518	∞	0.0253	0.0000	0.0000	0.0000	0.0000
cos 5°	0.5695	0.0468	0.3811	0.0210	0.0931	0.0044	0.0764	0.0021
cos 10°	0.2333	0.0370	0.0589	0.0134	0.1264	0.0080	0.0938	0.0036
cos 15°	0.0479	0.0253	—0.1038	0.0052	0.1380	0.0108	0.0911	0.0044
cos 20°	—0.0720	0.0133	—0.1932	—0.0023	0.1366	0.0124	0.0777	0.0045
cos 25°	—0.1531	0.0018	—0.2337	—0.0082	0.1265	0.0131	0.0587	0.0040
cos 30°	—0.2075	—0.0086	—0.2500	—0.0124	0.1106	0.0128	0.0372	0.0031
cos 35°	—0.2420	—0.0174	—0.2397	—0.0147	0.0909	0.0116	0.0157	0.0019
cos 40°	—0.2607	—0.0244	—0.2124	—0.0152	0.0689	0.0098	—0.0041	0.0006
cos 45°	—0.2664	—0.0294	—0.1733	—0.0141	0.0458	0.0074	—0.0210	—0.0007
cos 50°	—0.2613	—0.0324	—0.1265	—0.0117	0.0227	0.0047	—0.0341	—0.0018
cos 55°	—0.2472	—0.0334	—0.0758	—0.0083	0.0004	0.0018	—0.0430	—0.0027
cos 60°	—0.2253	—0.0325	—0.0247	—0.0043	—0.0202	—0.0011	—0.0473	—0.0032
cos 65°	—0.1972	—0.0299	0.0240	—0.0001	—0.0387	—0.0038	—0.0473	—0.0034
cos 70°	—0.1638	—0.0259	0.0676	0.0039	—0.0545	—0.0062	—0.0433	—0.0033
cos 75°	—0.1264	—0.0186	0.1038	0.0073	—0.0672	—0.0083	—0.0358	—0.0028
cos 80°	—0.0859	—0.0142	0.1309	0.0100	—0.0765	—0.0098	—0.0255	—0.0020
cos 85°	—0.0435	—0.0205	0.1477	0.0117	—0.0821	—0.0107	—0.0132	—0.0011
cos 90°	0.0000	0.0000	0 1534	0.0123	—0.0840	—0.0111	0.0000	0.0000

3. We define the function $f^{(s)}(x)$ by means of the equation

$$f^{(s)}(x)=\left(\frac{d^s}{d\theta^s}f(\cos\theta)\right)_{\theta=\arccos x} \qquad (s=0,1,\dots) \tag{3.1}$$

and call it the trigonometric derivative of order s of the function $f(x)$. We denote by $W_\gamma^{(2k)}(M_{2k};\ -1,\ 1)$ the class of functions satisfying the following conditions. (1) An arbitrary function $f(x)$ belonging to this class possesses continuous trigonometric derivatives $f^{(s)}(x)$, $(s = 0, 1, \dots)$ up to the order $(2k-1)$, inclusive, on the interval between the limits -1, 1 except at the point $x = 0$. (2) The $2k$-th trigonometric derivative of this function satisfies the inequality

$$|f^{(2k)}(x)|\leqslant M_{2k}; \tag{3.2}$$

(3) At the point $x = 0$, the function $f(x)$ and its trigonometric derivatives $f^{(s)}(x)$ can have discontinuities of the first kind:

$$2\gamma^{(s)}=f^{(s)}(+0)-f^{(s)}(-0) \qquad (s=0,1,\dots,2k) \tag{3.3}$$

The class of functions $W_\gamma^{(2k)}(M_{2k};\ -1,\ 1)$ is a type of generalization of the class $W^{(r)}(W;\ a,\ b)$ considered by Nikol'skii [12, 13].

Let us introduce the notations:

$$2\gamma_1^{(s)}=f^{(s)}(1)+f^{(s)}(-1), \qquad 2\gamma_2^{(s)}=f^{(s)}(1)-f^{(s)}(-1) \qquad (s=0,1,\dots,2k) \tag{3.4}$$

$$a_1^*=a_1, \qquad a_2^*=a_2, \qquad b_1^*=b_1, \qquad b_2^*=b_2 \tag{3.5}$$

$$a_{2m}^*=a_{2m}+\frac{4}{\pi}\sum_{s=1}^{k}\frac{(-1)^s}{(2m)^{2s}}\left((-1)^m\gamma^{(2s-1)}-\gamma_2^{(2s-1)}\right)$$

$$a_{2m-1}^*=a_{2m-1}-\frac{4}{\pi}\sum_{s=1}^{k}\frac{(-1)^s}{(2m-1)^{2s}}\left((-1)^m(2m-1)\gamma^{(2s-1)}+\gamma_1^{(2s-1)}\right)$$

$$b_{2m}^*=b_{2m}-\frac{4}{\pi}\sum_{s=1}^{k}\frac{(-1)^s}{(2m)^{2s-1}}\left((-1)^m\gamma^{2(s-1)}-\gamma_2^{2(s-1)}\right) \tag{3.6}$$

$$b_{2m-1}^*=b_{2m-1}-\frac{4}{\pi}\sum_{s=1}^{k}\frac{(-1)^s}{(2m-1)^{2s}}\left((-1)^m\gamma^{(2s-1)}-(2m-1)\gamma_1^{2(s-1)}\right)$$

where a_n and b_n are Fourier coefficients given by the formulas (1.4).

We introduce the number $N(n,\ k)$ by means of the defining equation:

$$N(n,k)=4(n+1)^{-2k}\left(1+\ln\frac{\pi}{2}+\frac{1}{2k}+\frac{1}{2n}+\ln n\right) \tag{3.7}$$

Theorem 1. Let $f(x)$ $W_{\gamma}^{(2k)}(M_{2k}; -1, 1)$, then the following two representations are valid for the function $f(x)$ on the interval between the limits -1, 1

$$f(x) = \frac{4}{\pi}\sum_{s=1}^{k}(-1)^s[\gamma_1^{(2s-1)}p_1^{(2s)}(x) - (\operatorname{sign} x)\gamma^{2(s-1)}q_1^{(2s-1)}(x^*) + \tag{3.8}$$

$$+\gamma_2^{(2s-1)}p_2^{(2s)}(x) - \gamma^{(2s-1)}p_2^{(2s)}(x^*)] + \frac{a_0}{2} + \sum_{m=1}^{n} a_m^* T_m(x) + r_n^{(1)}(x)$$

$$f(x) = -\frac{4}{\pi}\sum_{s=1}^{k}(-1)^s[\gamma^{(2s-1)}p_1^{(2s)}(x^*) + \gamma_1^{2(s-1)}q_1^{(2s-1)}(x) + \gamma_2^{2(s-1)}q_2^{(2s-1)}(x) +$$

$$+(\operatorname{sign} x)\gamma^{2(s-1)}q_2^{(2s-1)}(x^*)] + \sum_{m=1}^{n} b_m^* U_m(x) + r_n^{(2)}(x) \tag{3.9}$$

where $r_n^{(1)}(x)$ and $r_n^{(2)}(x)$ satisfy the inequalities

$$|r_n^{(1)}(x)| < M_{2k}N(n,k), \qquad |r_n^{(2)}(x)| < M_{2k}N(n,k) \tag{3.10}$$

Proof. Let us apply the formula for integration by parts $2k$ times to the integrals occurring in the equations (1.4). We then obtain the relations (3.6) where

$$a_n^* = \frac{(-1)^k}{n^{2k}}a_n^{(2k)}, \qquad a_n^{(2k)} = \frac{2}{\pi}\int_{-1}^{1} f^{(2k)}(x)\,T_n(x)\frac{dx}{\sqrt{1-x^2}}$$

$$b_n^* = \frac{(-1)^k}{n^{2k}}b_n^{(2k)}, \qquad b_n^{(2k)} = \frac{2}{\pi}\int_{-1}^{1} f^{(2k)}(x)\,U_n(x)\frac{dx}{\sqrt{1-x^2}} \tag{3.11}$$

Substituting into the series (1.5) and (1.6) for a_n and b_n their expressions from (3.6), we obtain the equations (3.8) and (3.9) where

$$r_n^{(1)}(x) = \sum_{m=n+1}^{\infty} a_m^* T_m(x), \qquad r_n^{(2)}(x) = \sum_{m=n+1}^{\infty} b_m^* U_m(x)$$

The formulas (3.11) can be used for the determination a_n and b_n, if one knows the derivative $f^{(2k)}(x)$.

We shall prove that $r_n^{(1)}(x)$ and $r_n^{(2)}(x)$ satisfy the equation (3.10). If we introduce into our consideration the sums

$$\sigma_m^{(1)}(x) = \sum_{\nu=1}^{m} a_\nu^{(2k)} T_\nu(x), \qquad \sigma_m^{(2)}(x) = \sum_{\nu=1}^{m} b_\nu^{(2k)} U_\nu(x),$$

then we obtain the following relation for $r_n^{(s)}(x)$. $s = 1, 2$:

$$|r_n^{(s)}(x)| = \left| \sum_{m=n+1}^{\infty} \frac{\sigma_m^{(s)}(x) - \sigma_{m-1}^{(s)}(x)}{m^{2k}} \right| = \qquad (s=1,2)$$

$$= \left| -\frac{\sigma_n^{(s)}(x)}{(n+1)^{2k}} + \sum_{m=n+1}^{\infty} \left(\frac{1}{m^{2k}} - \frac{1}{(m+1)^{2k}} \right) \sigma_m^{(s)}(x) \right| \tag{3.12}$$

The function $f(x)$ $W_\gamma^{2k}(M_{2k};\ -1,\ 1)$ and hence the inequality (3.2) can be applied. It is, therefore, easy to show that

$$|\sigma_m^{(s)}(x)| < 2M_{2k}\left(1 + \ln\frac{\pi}{2} + \ln m\right) \qquad (s=1,2) \tag{3.13}$$

Let us estimate each term of the series (3.12) with the aid of (3.13), and by means of the sequence of inequalities

$$\ln\frac{m+1}{m} < \frac{1}{m}, \qquad \sum_{m=n+1}^{\infty} \frac{1}{(m+1)^{2k+1}} < \frac{1}{2k(n+1)^{2k}}$$

We obtain the result

$$|r_n^{(s)}(x)| < M_{2k}\frac{2}{(n+1)^{2k}}\left(2 + 2\ln\frac{\pi n}{2} + \frac{1}{n} + \frac{1}{k}\right) = M_{2k}N(n,k)$$
$$(s=1,2)$$

which was to be proved.

4. The results of the preceding section make it possible to obtain approximate formulas for the evaluation of the integrals $J(x)$ and $I(x)$.

Theorem 2. Let $f(x)$ $W_\gamma^{(2k)}(M_{2k};\ -1,\ 1)$, then

$$J(x) \approx J_n^{(k)}(x) = \frac{4}{\pi}\sum_{s=1}^{k}(-1)^s[\gamma_1^{2(s-1)}p_1^{(2s-1)}(x) + (\operatorname{sign} x)\gamma^{(2s-1)}q_1^{(2s)}(x^*) +$$
$$+ \gamma_2^{2(s-1)}p_2^{(2s-1)}(x) - \gamma^{2(s-1)}p_2^{(2s-1)}(x^*)] - \sum_{m=1}^{n} b_m^* T_m(x) \tag{4.1}$$

$$I(x) \approx I_n^{(k)}(x) = \frac{4}{\pi}\sum_{s=1}^{k}(-1)^s[-\gamma^{2(s-1)}p_1^{(2s-1)}(x^*) + \gamma_1^{(2s-1)}q_1^{(2s)}(x) -$$
$$+ \gamma_2^{(2s-1)}q_2^{(2s)}(x) + (\operatorname{sign} x)\gamma^{(2s-1)}q_2^{(2s)}(x^*)] + \sum_{m=1}^{n} a_m^* U_m(x) \tag{4.2}$$

wherein

$$|J(x) - J_n^{(k)}(x)| < M_{2k}N(n,k), \qquad |I(x) - I_n^{(k)}(x)| < M_{2k}N(n,k) \tag{4.3}$$

Proof. Let us substitute for $f(x)$ its expressions from (3.8) and (3.9) into the integrals $J(x)$ and $I(x)$, respectively. Making use of the equations (1.1), (2.4) and (2.5), we obtain

$$J(x) = J_n^{(k)}(x) + r_n^{(3)}(x), \qquad I(x) = I_n^{(k)}(x) + r^{(4)}(x)$$

where

$$r_n^{(3)}(x) = -\sum_{m=n+1}^{\infty} b_m^* T_m(x), \qquad r_n^{(4)}(x) = \sum_{m=n+1}^{\infty} a_m U_m(x)$$

In a manner entirely analogous to the one used in the preceding section for the estimate of $r_n^{(1)}(x)$ and $r_n^{(2)}(x)$, we can show that

$$|r_n^{(3)}(x)| < M_{2k}N(n,k), \qquad |r_n^{(4)}(x)| < M_{2k}N(n,k)$$

Comparing these inequalities with the expression for $J(x)$ and $I(x)$ obtained above, we establish the validity of formulas (4.1), (4.2) and (4.3).

We call attention to three particular cases when the formulas (4.1) and (4.2) are simplified.

(1) The function $f(x)$ and its derivative are continuous at the point $x = 0$.

$$J(x) \approx J_n^{(k)}(x) = \frac{4}{\pi}\sum_{s=1}^{k}(-1)^s[\gamma_1^{2(s-1)}p_1^{(2s-1)}(x) + \gamma_2^{2(s-1)}p_2^{(2s-1)}(x)] -$$

$$-\sum_{m=1}^{n} b_m^* T_m(x), \qquad |J(x) - J_n^{(k)}(x)| < M_{2k}N(n, k) \tag{4.4}$$

$$I(x) \approx I_n^{(k)}(x) = \frac{4}{\pi}\sum_{s=1}^{k}(-1)^s[\gamma_1^{(2s-1)}q_1^{(2s)}(x) + \gamma_2^{(2s-1)}q_2^{(2s)}(x)] +$$

$$+\sum_{m=1}^{n} a_m^* U_m(x), \qquad |I(x) - I_n^{(k)}(x)| < M_{2k}N(n, k) \tag{4.5}$$

(2) The function $f(x)$ is even, i.e. $f(-x) = f(x)$:

$$J(x) \approx J_n^{(k)}(x) = \frac{4}{\pi}\sum_{s=1}^{k}(-1)^s[\gamma_1^{2(s-1)}p_1^{(2s-1)}(x) + (\operatorname{sign} x)\gamma^{(2s-1)}q_1^{(2s)}(x^*)] -$$

$$-\sum_{m=1}^{n} b_{2m-1}^* T_{2m-1}(x), \qquad |J(x) - J_n^{(k)}(x)| < M_{2k}N(2n-1, k) \tag{4.6}$$

$$I(x) \approx I_n^{(k)}(x) = \frac{4}{\pi}\sum_{s=1}^{k}(-1)^s[\gamma_2^{(2s-1)}q_2^{(2s)}(x) + (\operatorname{sign} x)\gamma^{(2s-1)}q_2^{(2s)}(x^*)] +$$

$$+\sum_{m=1}^{n} a_{2m}^* U_{2m}(x), \qquad |I(x) - I_n^{(k)}(x)| < M_{2k}N(2n, k) \tag{4.7}$$

(3) The function $f(x)$ is odd, i.e. $f(-x) = -f(x)$:

$$J(x) \approx J_n^{(k)}(x) = \frac{4}{\pi} \sum_{s=1}^{k} (-1)^s [\gamma_2^{2(s-1)} p_2^{(2s-1)}(x) - \gamma^{2(s-1)} p_2^{(2s-1)}(x^*)] -$$

$$- \sum_{m=1}^{n} b_{2m}^* T_{2m}(x), \qquad |J(x) - J_n^{(k)}(x)| < M_{2k} N(2n, k) \qquad (4.8)$$

$$I(x) \approx I_n^{(k)}(x) = \frac{4}{\pi} \sum_{s=1}^{k} (-1)^s [-\gamma^{2(s-1)} p_1^{(2s-1)}(x^*) + \gamma_1^{(2s-1)} q_1^{(2s)}(x)] +$$

$$+ \sum_{m=1}^{n} a_{2m-1}^* U_{2m-1}(x), \qquad |I(x) - I_n^{(k)}(x)| < M_{2k} N(2n-1, k) \quad (4.9)$$

If the function $f(x)$ is even, then

$$J(-x) = -J(x), \qquad I(-x) = -I(x) \qquad (4.10)$$

If the function $f(x)$ is odd, then

$$J(-x) = J(x), \qquad I(-x) = I(x) \qquad (4.11)$$

In the nature of an example illustrating the presented method, we consider the integral

$$I(x) = \frac{\sqrt{1-x^2}}{\pi} \int_{-1}^{1} \frac{{}^1/_4 \pi\, |t|}{t-x} \frac{dt}{\sqrt{1-t^2}} \qquad (-1 \leqslant x \leqslant 1) \qquad (4.12)$$

Here the function $f(x) = \pi |x| / 4$ is even. Hence, for the evaluation of this integral one can make use of the particular case (4.7) of the formula (4.2). Using formulas (3.1) to (3.6) we find

$$f^{(2s-1)}(x) = (-1)^s \frac{\pi}{4} (\operatorname{sign} x) \sqrt{1-x^2}, \qquad f^{(2s)}(x) = (-1)^s \frac{\pi}{4} |x|$$

$$\gamma_2^{(2s-1)} = 0, \qquad \gamma^{(2s-1)} = (-1)^s \frac{\pi}{4}, \qquad M_{2k} = \frac{\pi}{4}$$

$$a_{2m} = \frac{(-1)^{m-1}}{4m^2-1} \quad (m = 1, 2, \ldots); \qquad a_2^* = \frac{1}{3}, \quad a_{2m}^* = \frac{(-1)^{m-1}}{(2m)^{2k}(4m^2-1)} \quad (m = 2, 3, \ldots)$$

Substituting the obtained quantities into the formula (4.7), we obtain

$$I(x) \approx I_n^{(k)}(x) = (\operatorname{sign} x) \sum_{s=1}^{k} q_2^{(2s)}(x^*) + \frac{1}{3} U_2(x) + \sum_{m=2}^{n} \frac{(-1)^{m-1}}{(2m)^{2k}(4m^2-1)} U_{2m}(x) \qquad (4.13)$$

$$|I(x) - I_n^{(k)}(x)| < \frac{\pi}{4} N(2n, k) \qquad (4.14)$$

The formula (4.13) makes it possible, with the use of the estimate (4.4), to evaluate the integral (4.12) with an arbitrarily prescribed degree of accuracy. On the other hand, with the aid of formula (1.3), or by direct integrations, one can find the exact value of the integral (4.12) in terms of elementary functions

$$I(x) = x \operatorname{Arth} \sqrt{1-x^2} = x \operatorname{Arch} \frac{1}{x}$$

Since the considered integral is an odd function, one can restrict its evaluation to the interval between the limits 0 and 1 and compute the approximate value of the integral at various points of the interval between the limits 0 and 1 and compare these results with the exact values of the integral. We obtain the following results:

$x=0.0$	0.1	cos 80°	0.2	0.3	cos 70°	0.4	0.5	0.6
$I(x)=0.0000$	0.1497	0.2115	0.2292	0.2811	0.2968	0.3134	0.3292	0.3296
$I_2^2(x)=0.0000$	0.1497	0.2116	0.2292	0.2811	0.2967	0.3134	0.3292	0.3295
$x=\cos 50°$	0.7	cos 40°	0.8	cos 30°	0.9	cos 20°	cos 10°	1.0
$I(x)=0.3248$	0.3135	0.2922	0.2773	0.2379	0.2102	0.1674	0.0864	0.0000
$I_2^2(x)=0.3248$	0.3135	0.2922	0.2774	0.2380	0.2103	0.1673	0.0863	0.0000

From this it can be seen that the approximate values of the integral (4.12), computed by means of the formulas (4.13) for $n = 2$ and $k = 2$, differ from the exact values only by one in the fourth decimal place. The estimate (4.14) in this case yields

$$|I(x) - I_2^{(2)}(x)| < 0.0162$$

i.e. the formula (4.13) is actually more precise than the estimate (4.14) indicates.

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